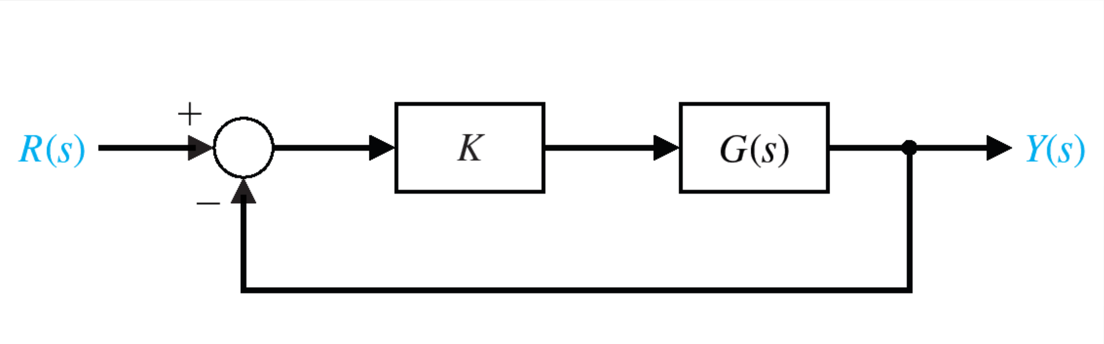
Control Systems

Lecture: 5

Topics Covered

Root Locus

What is Root Locus ?



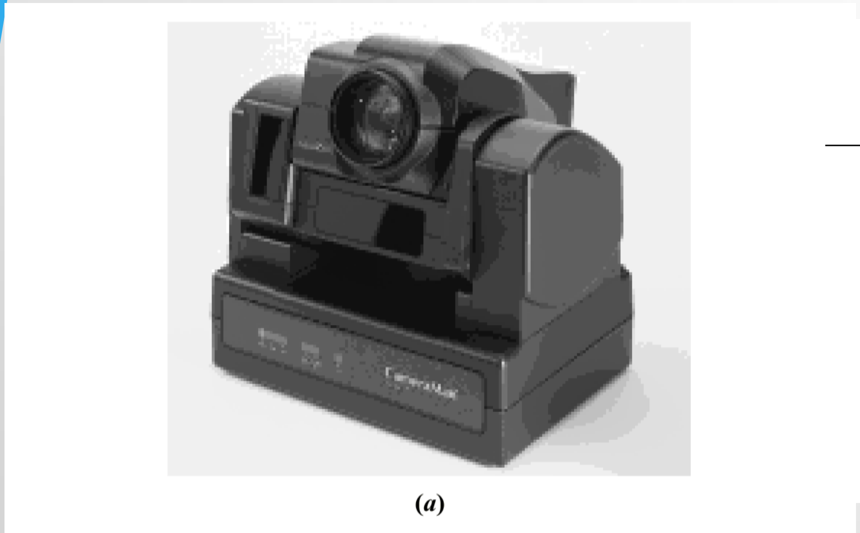
The characteristic equation of the closed-loop system is

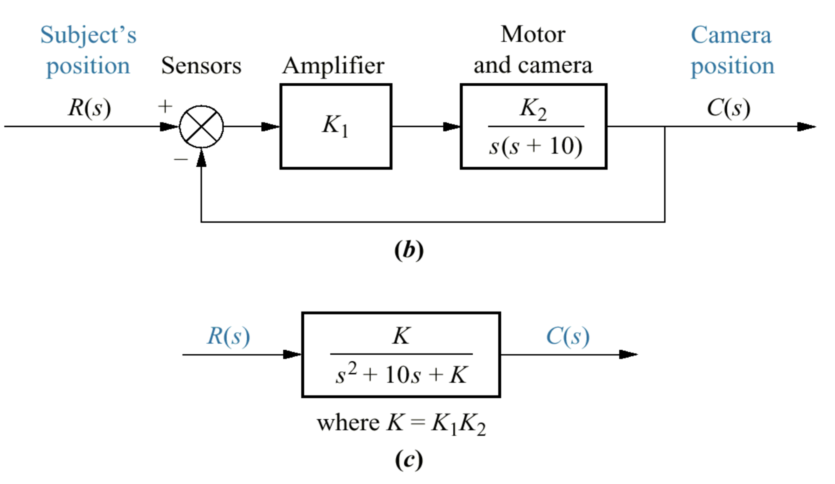
$$1 + K\,G(s) = 0$$

The root locus is essentially the trajectories of roots of the characteristic equation as the parameter *K* is varied from 0 to infinity.

A simple example

A camera control system:





How the dynamics of the camera changes as *K* is varied ?

A simple example (cont.) : pole locations

K	Pole 1	Pole 2
0	-10	0
5	-9.47	-0.53
10	-8.87	-1.13
15	-8.16	-1.84
20	-7.24	-2.76
25	-5	-5
30	$-5 + j2.24$	$-5 - j2.24$
35	$-5 + j3.16$	$-5 - j3.16$
40	$-5 + j3.87$	$-5 - j3.87$
45	$-5 + j4.47$	$-5 - j4.47$
50	$-5 + j5$	$-5 - j5$

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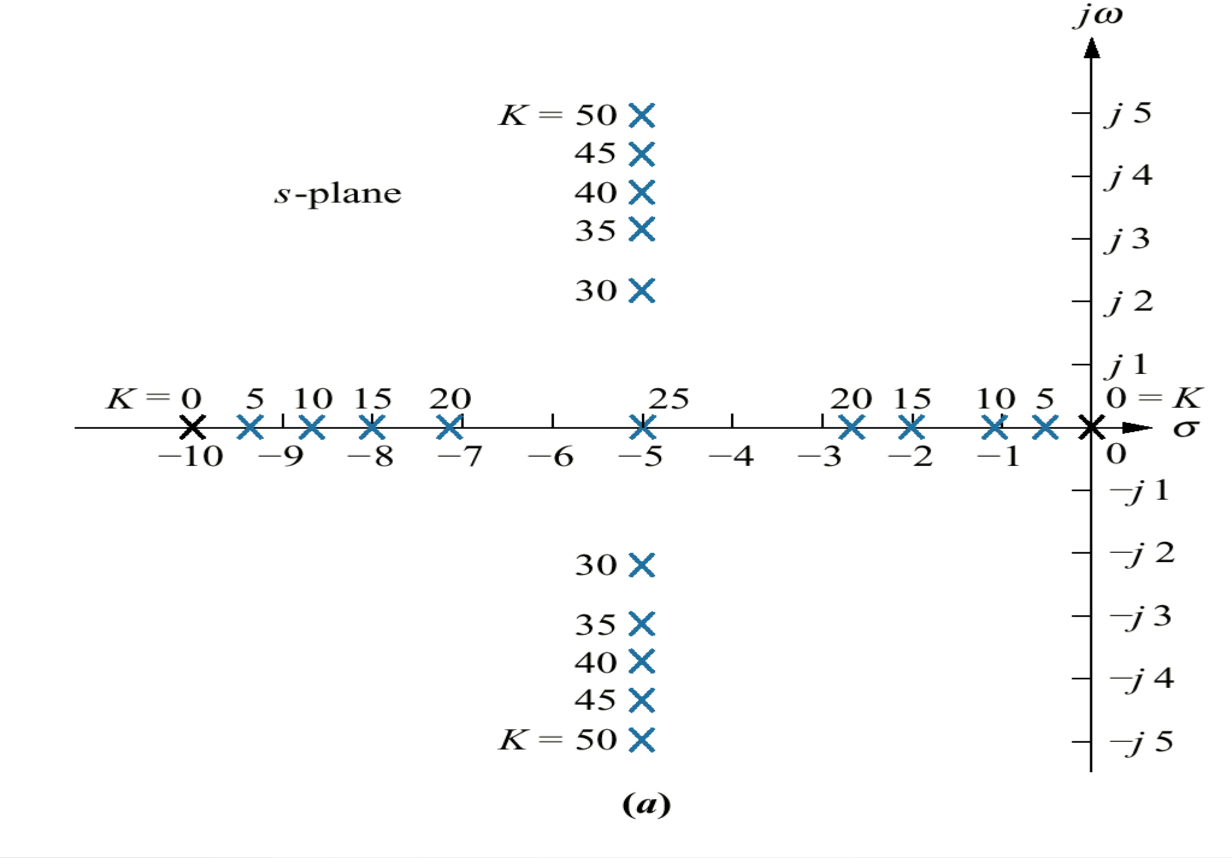
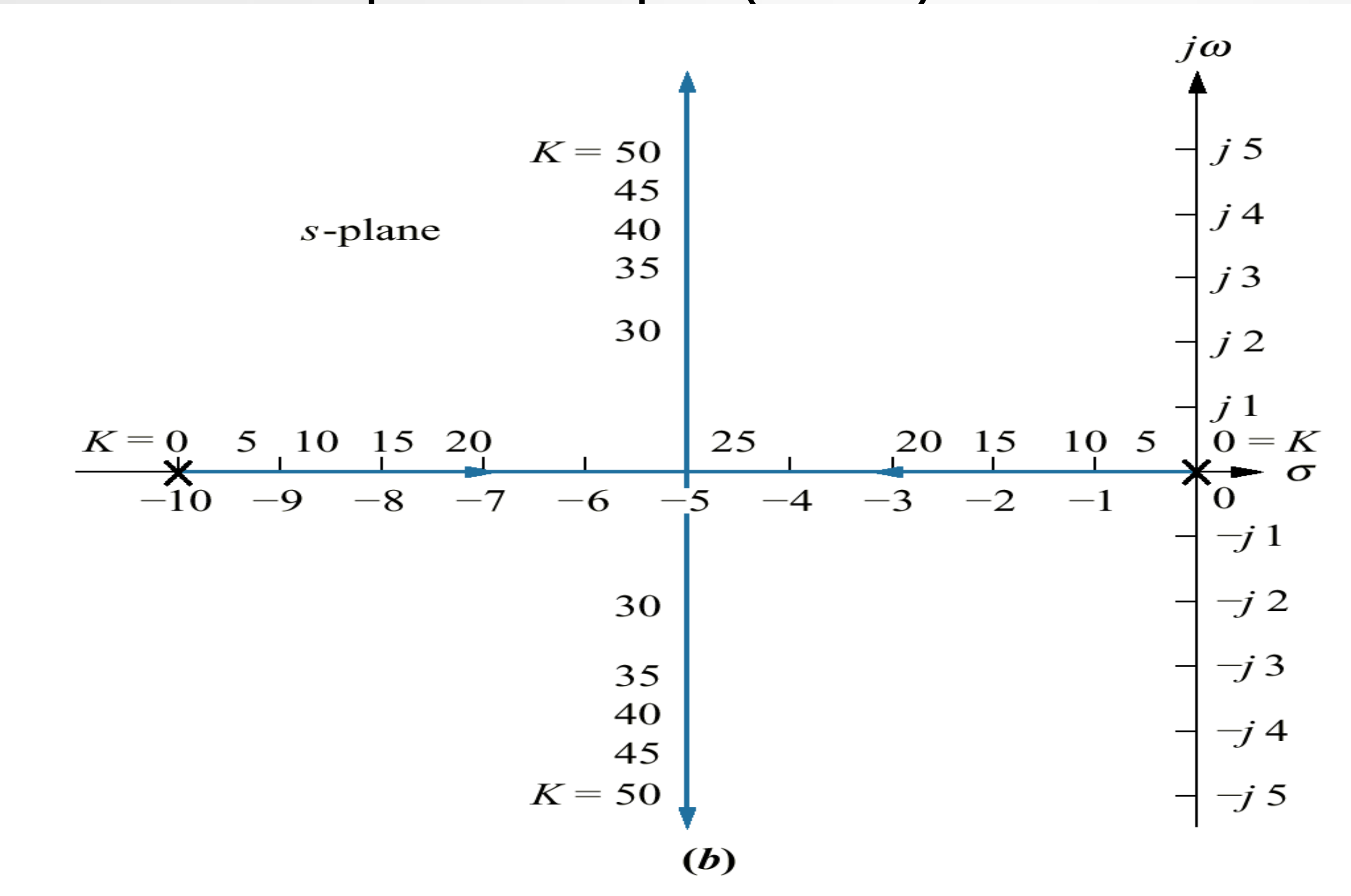


(a) Pole plots from the table.

A simple example (cont.) : Root Locus



(b) Root locus.

The Root Locus Method (cont.)

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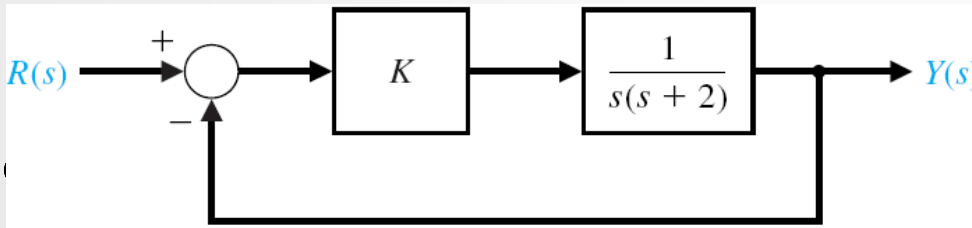


$$\Delta(s) = 1 + KG(s) = 1 + \frac{K}{s(s+2)} = 0$$

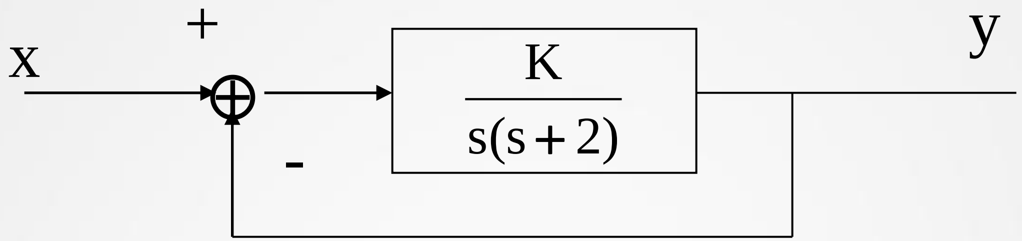
- The characteristic equation is:

$$\Delta(s) = s^2 + 2s + K = s^2 + 2\zeta\omega_n s + \omega_n^2 = 0$$

The locus of the roots as the gain K is varied is found by requiring :

$$|G(s)| = \left|\frac{K}{s(s+2)}\right| = 1 \qquad \text{and} \qquad \angle G(s) = \pm 180, \pm 540, \ldots$$

Introduction

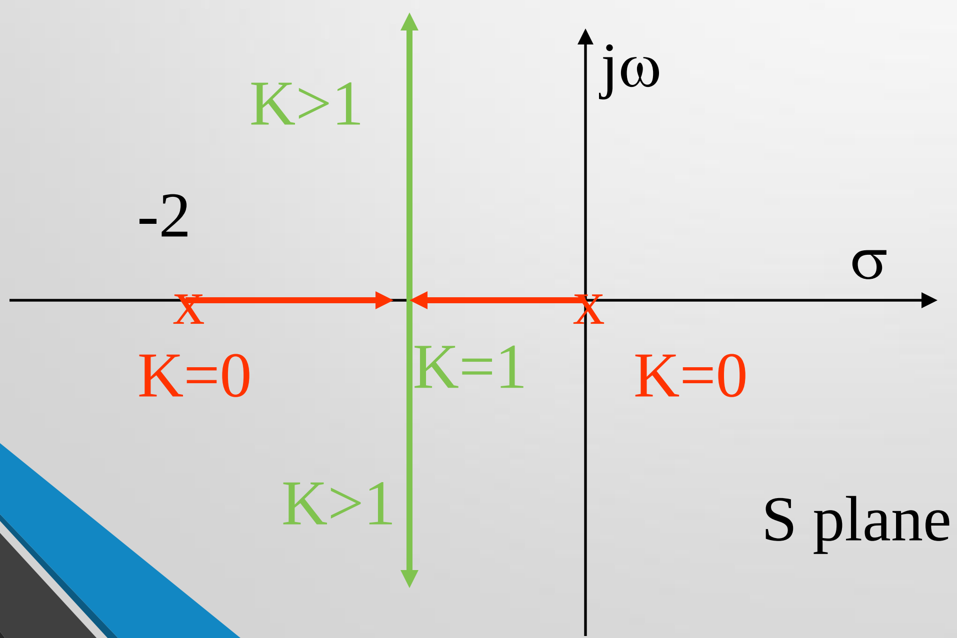


$$\frac{Y(s)}{X(s)} = \frac{G(s)}{1+G(s)} = \frac{K}{s^2+2s+K}$$

$$s_1, s_2 = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2-1}$$

For $\zeta < 1$ we know that $\theta = \cos^{-1}\zeta$

Characteristic equation $s^2+2s+K=0$



$0 < K < 1$

$\text{Roots} = -1 \pm \sqrt{1-K}$

$1 < K < \infty$

$\text{Roots} = -1 \pm j\sqrt{K-1}$

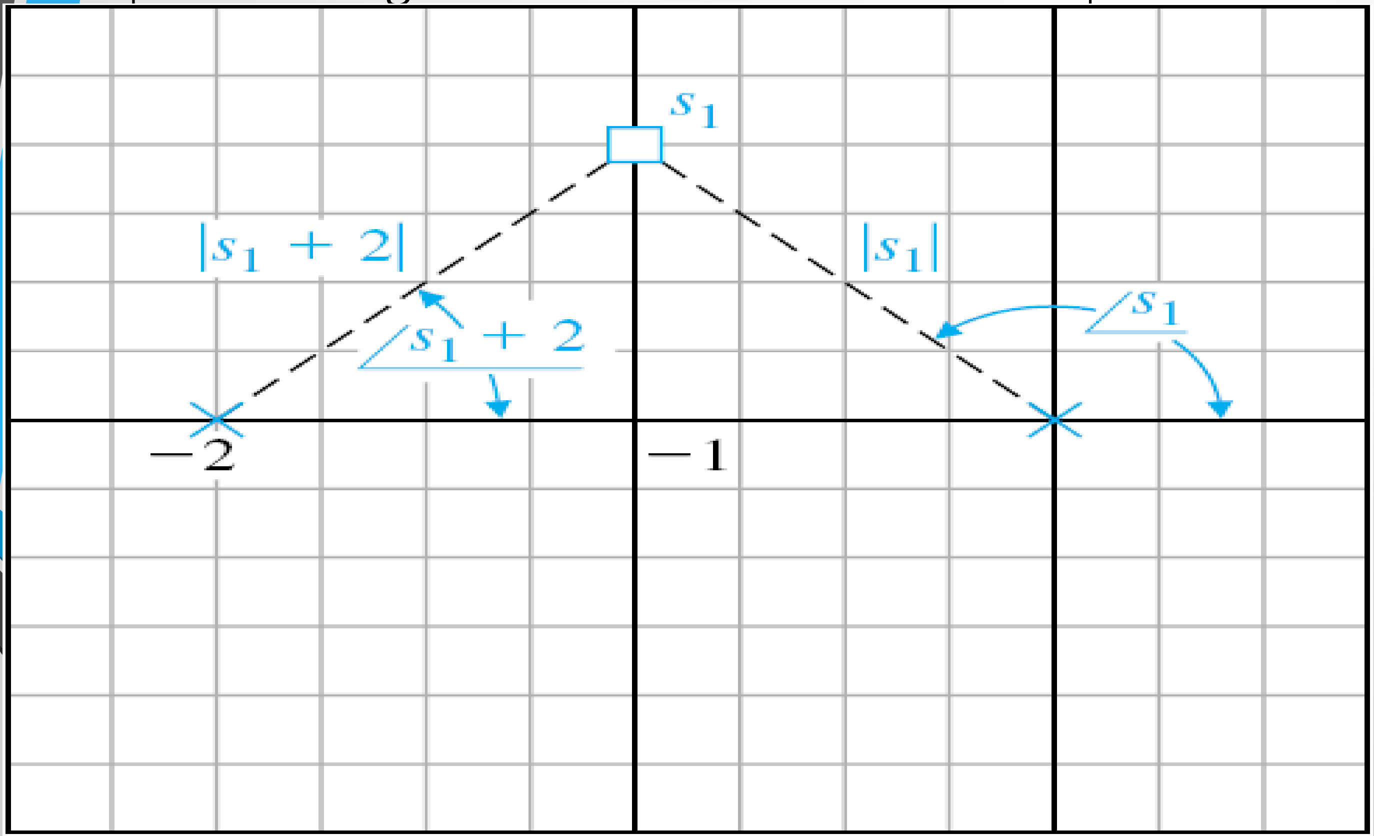
The Root Locus Method (cont.)

$$\angle\left(\frac{K}{s(s+2)}\right)\Bigg|_{s=s_1} = \frac{K}{|s_1||s_1+2|} = 1$$

$$\Rightarrow K = |s_1||s_1+2|$$

s_1 : the magnitude of the vector from the origin to s_1

$s_1 + 2$: the magnitude of the vector from - 2 to s_1



- Example:
- As shown below, at a root s_1, the angles are

The Root Locus Method (cont.)

- The magnitude and angle requirements for the root locus are:

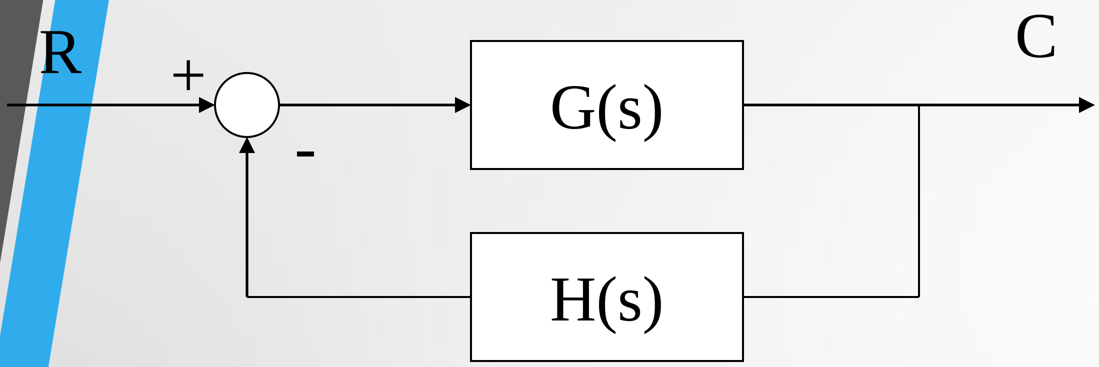
$$\left|F(s)\right| = \frac{K\left|s+z_1\right|\left|s+z_2\right|\ldots}{\left|s+p_1\right|\left|s+p_2\right|\ldots} = 1$$

$$\angle F(s) = \angle(s+z_1) + \angle(s+z_2) + \ldots - \left[\angle(s+p_1) + \angle(s+p_2) + \ldots\right] = 180 \pm q360$$

q : an integer

- The magnitude requirement enables us to determine the value of K for a given root location s_1.
- All angles are measured in a counterclockwise direction from a horizontal line.

Root locus



$$F = \frac{C}{R} = \frac{G}{1+GH}$$

$$GH = \frac{KN(s)}{D(s)} = \frac{K(s^m + a_{m-1}s^{m-1} + ...a_0)}{s^n + b_{n-1} + ...b_0}$$

Open loop transfer function

$$m < n$$

$$F = \frac{G}{1+KN/D} = \frac{GD}{D+KN}$$

Closed loop transfer function

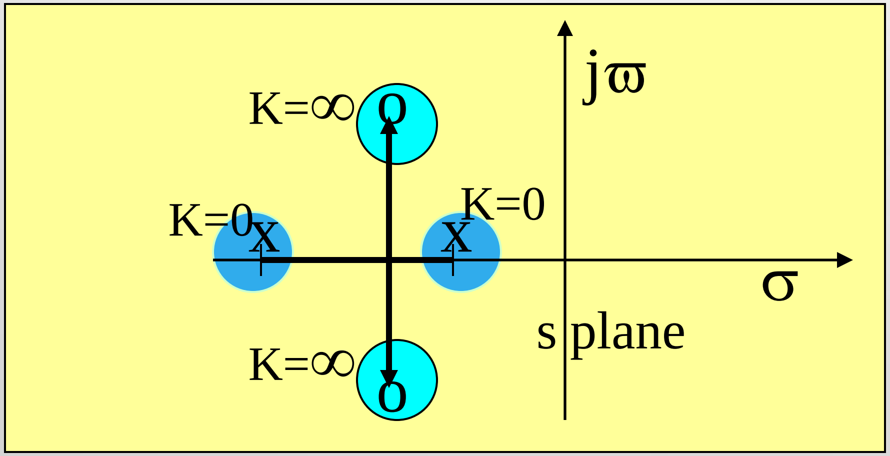
The poles of the closed loop are the roots of
The characteristic equation

$$D(s) + KN(s) = 0$$

Root locus (Evans)

$$P(s) = D(s) + KN(s) = 0 \qquad K \geq 0$$

- Root locus in the s plane are dependent on K
- If K=0 then the roots of P(s) are those of D(s): Poles of GH(s)
- If K=∞ then the roots of P(s) are those of N(s): Zeros of GH(s)



Root locus example

$$GH = \frac{K(s+1)}{s(s+2)}$$

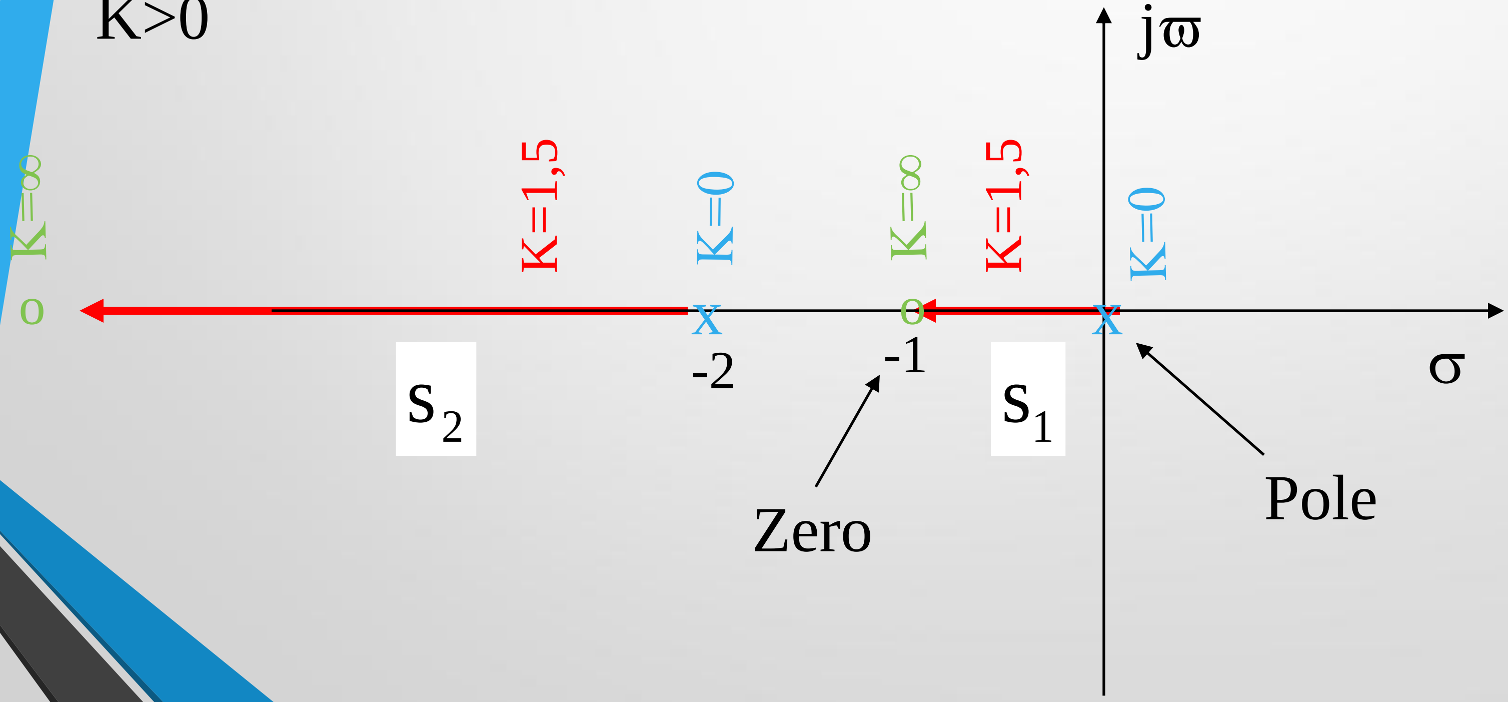
H=1

$$s_1 = -\frac{2+K}{2} + \sqrt{1+K^2/4}$$

$$s_2 = -\frac{2+K}{2} - \sqrt{1+K^2/4}$$

$$F(s) = \frac{K(s+1)}{s^2 + s(K+2) + K}$$

K>0



Root locus example

Any information from Rooth ?

$$F(s) = \frac{K(s+1)}{s^2 + s(K+2) + K}$$

$$s^2 + s(K+2) + K$$

s^2	1	K
s^1	$K+2$	0
s^0	K	0

As K>0

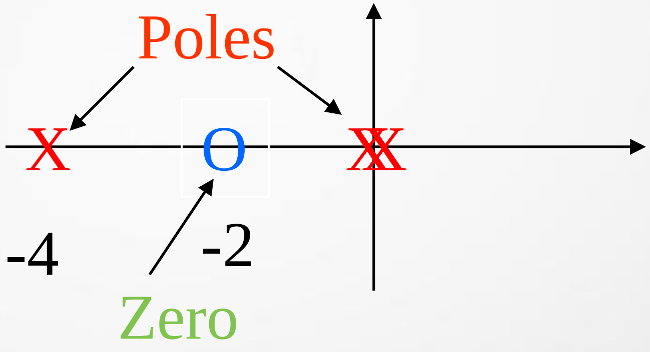
Rules for plotting root loci/loca

- Rule 1:

Number of loci: number of poles of the open loop transfer Function (the order of the characteristic equation)

$$GH = \frac{K(s+2)}{s^2(s+4)}$$

Three loci (branches)



- Rule 2:

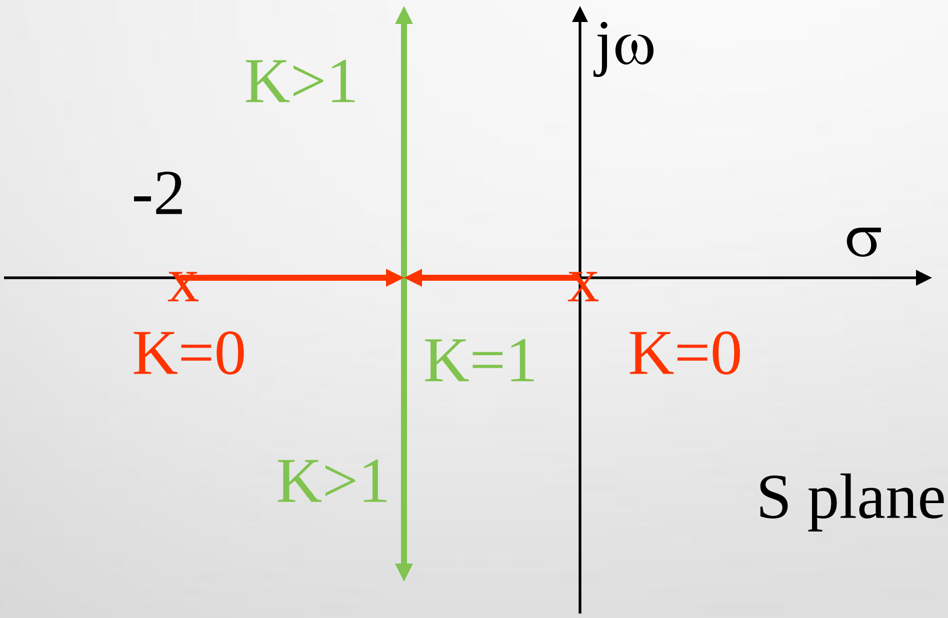
Each locus starts at an open-loop pole when K=0 and finishes Either at an open-loop zero or infinity when k= infinity

Problem: three poles and one zero ?

Rules for plotting root loci/loca

- Rule 3

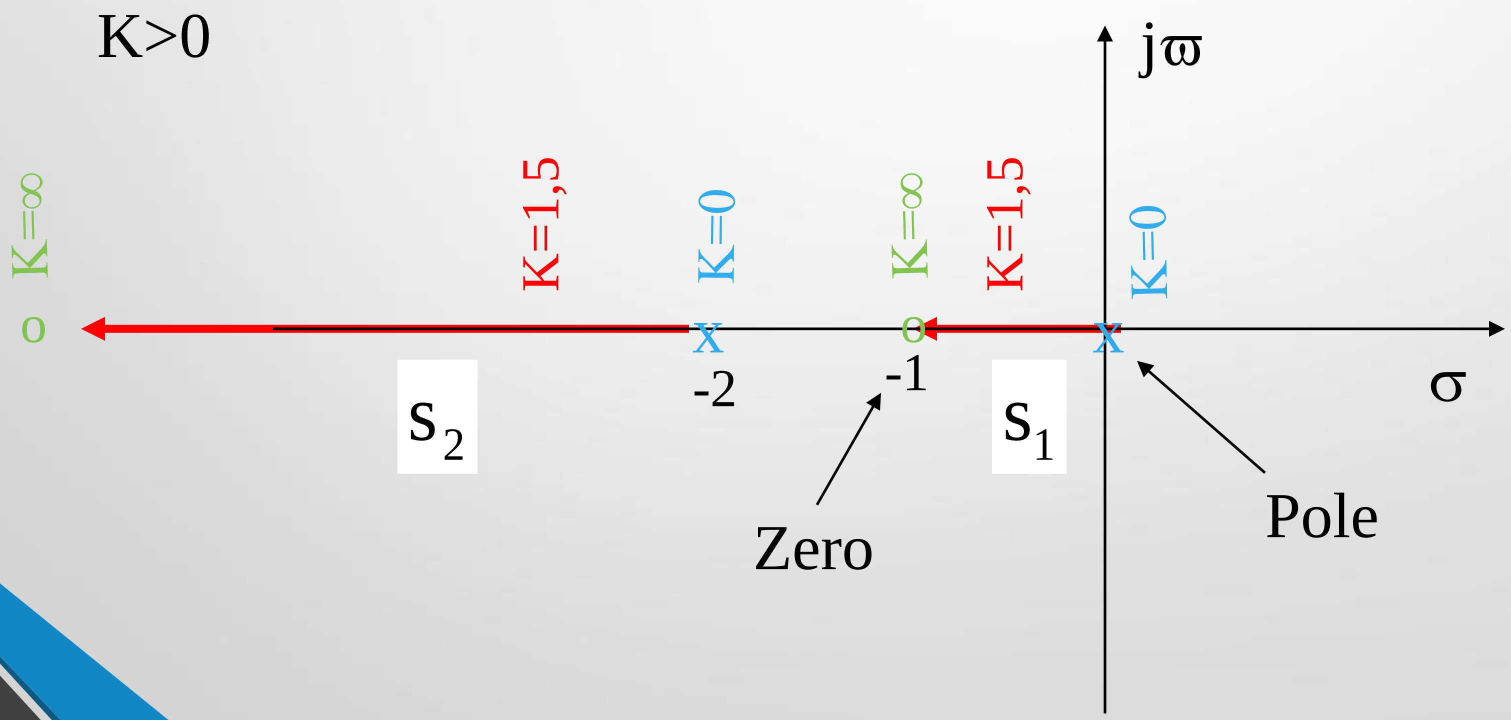
Loci either move along the real axis or occur as complex Conjugate pairs of loci



Rules for plotting root loci/loca

- Rule 4

A point on the real axis is part of the locus if the number of Poles and zeros to the right of the point concerned is odd for K>0

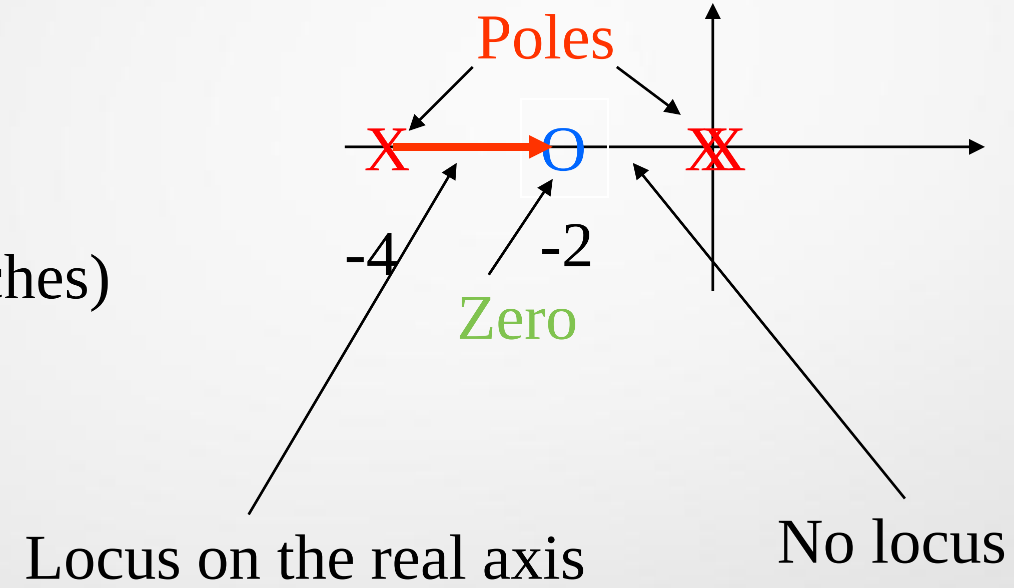


Rules for plotting root loci/loca

Example

$$GH = \frac{K(s+2)}{s^2(s+4)}$$

Three loci (branches)



Rules for plotting root loci/loca

•Rule 5:
When the locus is far enough from the open-loop poles and zeros,
It becomes asymptotic to lines making angles to the real axis
Given by: (n poles, m zeros of open-loop)

$$\Phi = \frac{(2L+1)180}{n-m}$$

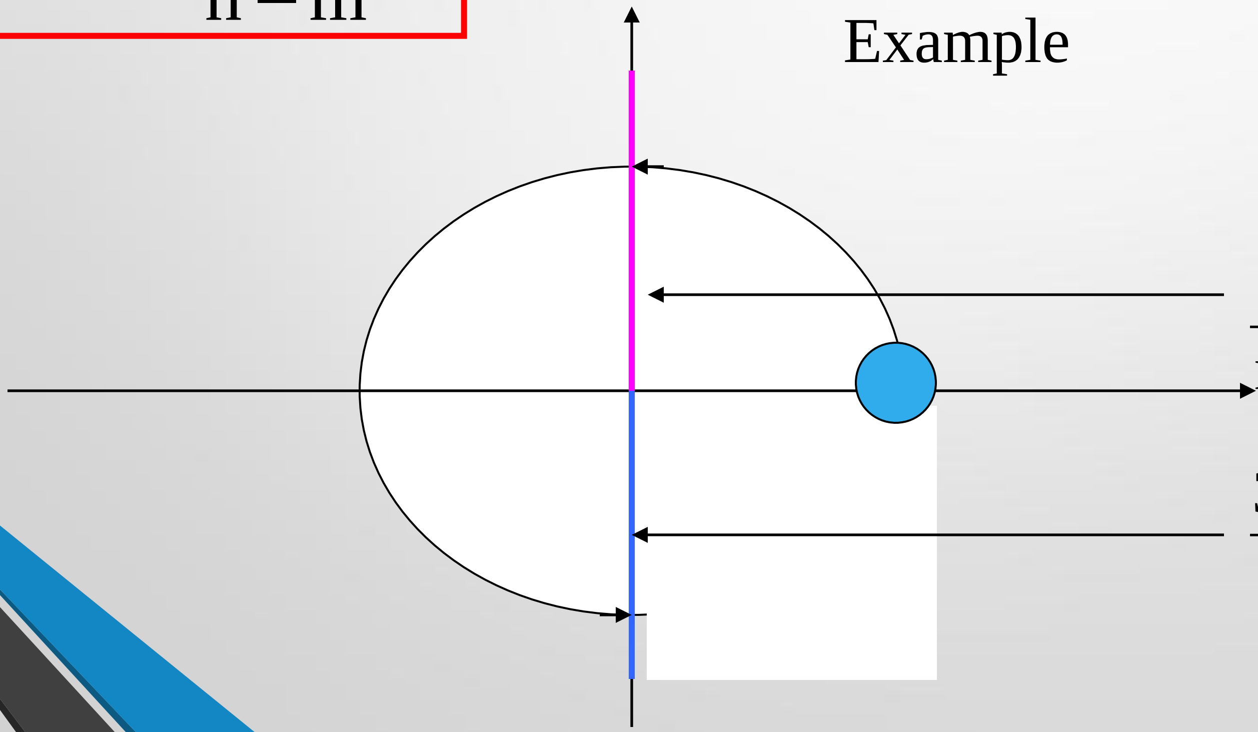
There are n-m asymptotes

L=0,1,2,3…..,(n-m-1)

Example

$$G(s) = \frac{K}{s(s+2)}$$

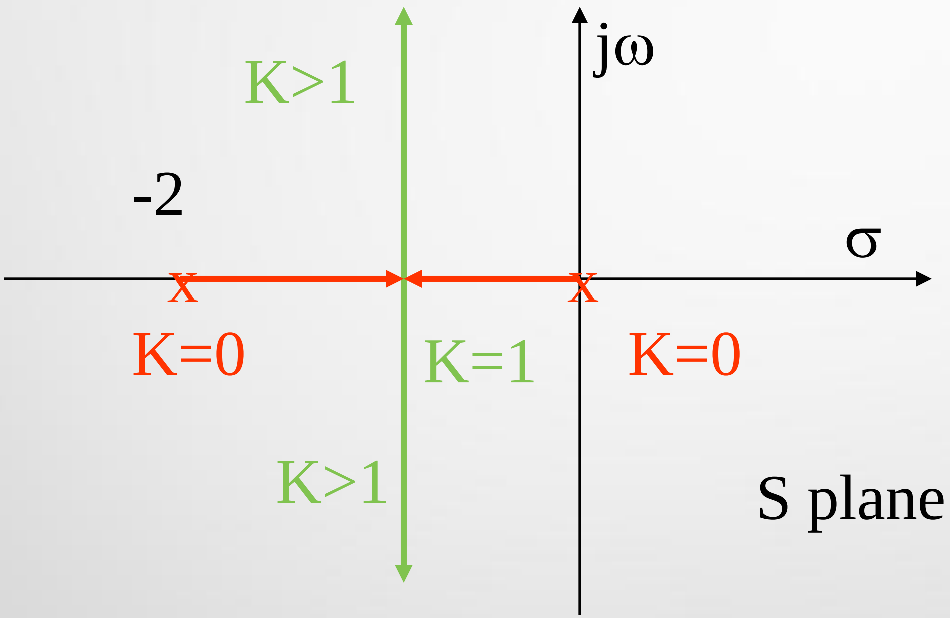
$$m = 0, n = 2$$



$$\frac{180}{n-m}(L=0) = 90$$

$$\frac{3*180}{n-m}(L=1) = 270$$

Rules for plotting root loci/loca



Rules for plotting root loci/loca

- Rule 6 :Intersection of asymptotes with the real axis

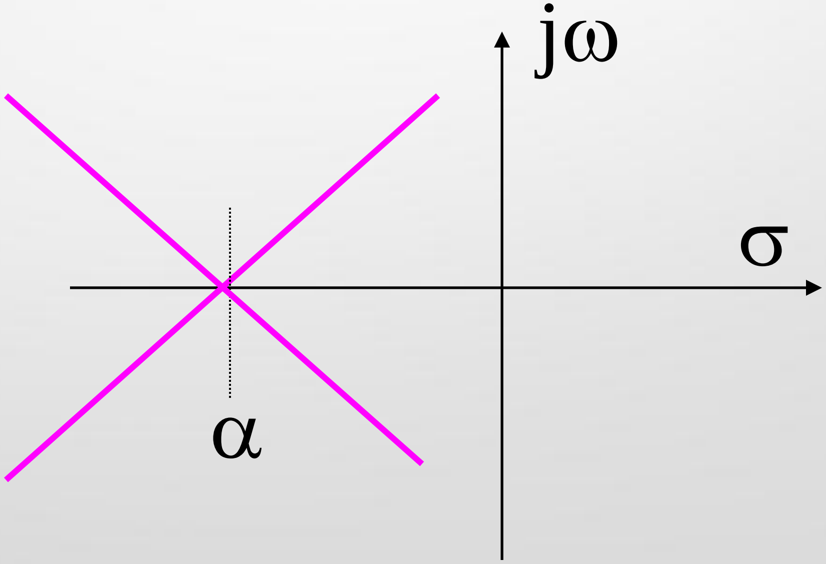
The asymptote intersect the real axis at a point α given by

$$\alpha = \frac{\sum_{i=1}^{n} p_i - \sum_{i=1}^{m} z_i}{n - m}$$

$$n - m = 4$$

$$p_i = \text{pole}$$

$$z_i = \text{zero}$$

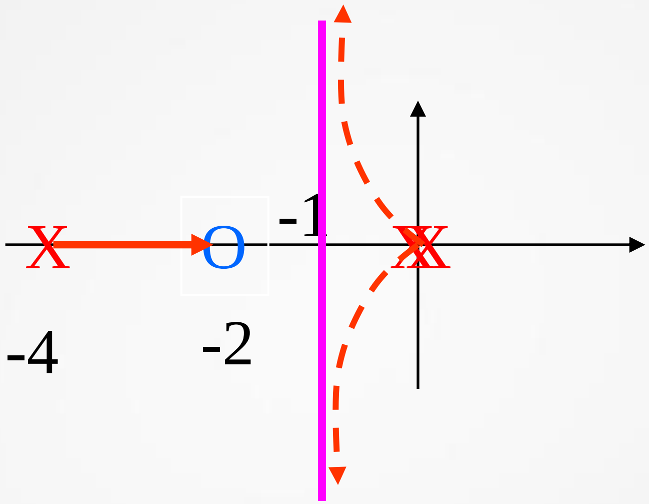


Rules for plotting root loci/loca

- Example

$$GH = \frac{K(s+2)}{s^2(s+4)}$$

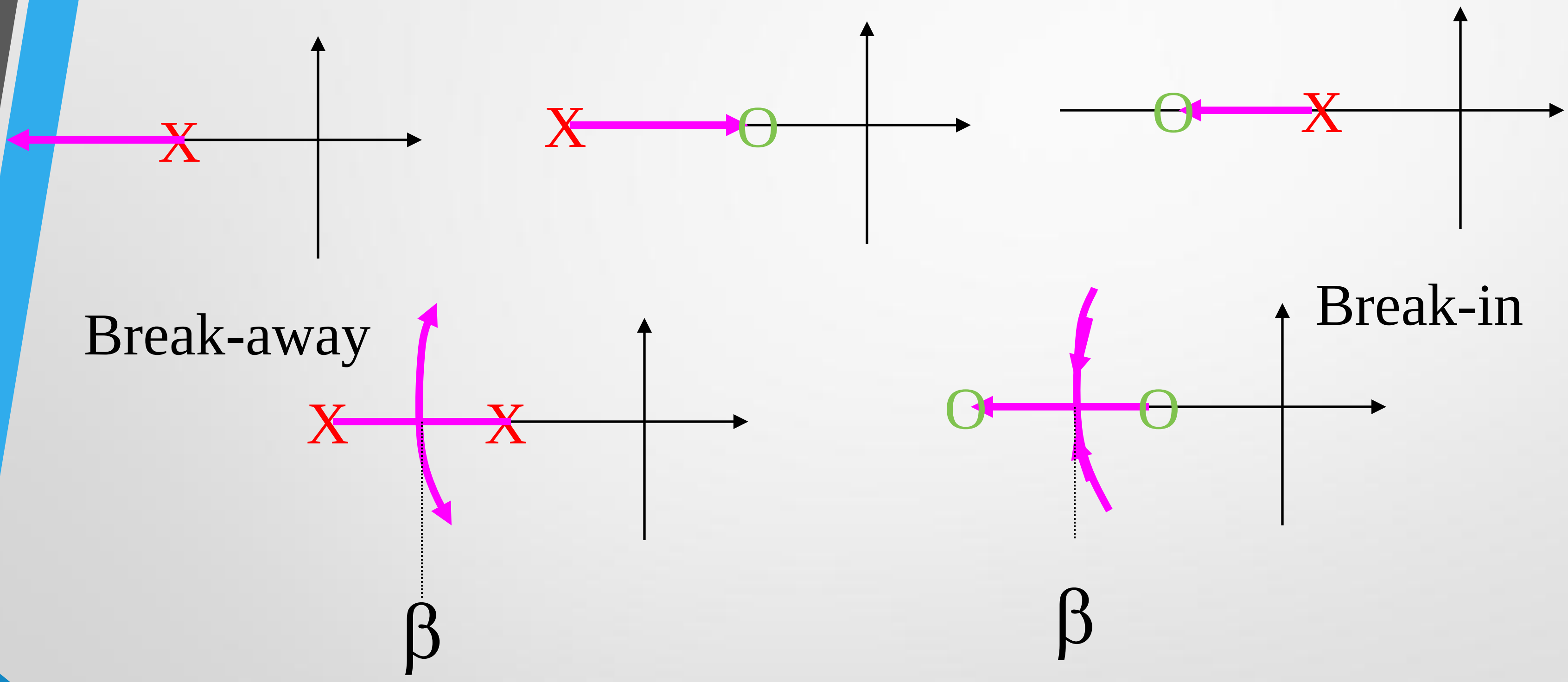
$$\alpha = \frac{-4+2}{2} = -1$$



Rules for plotting root loci/loca

- Rule 7

The break-away point between two poles, or break-in point Between two zero β is given by:



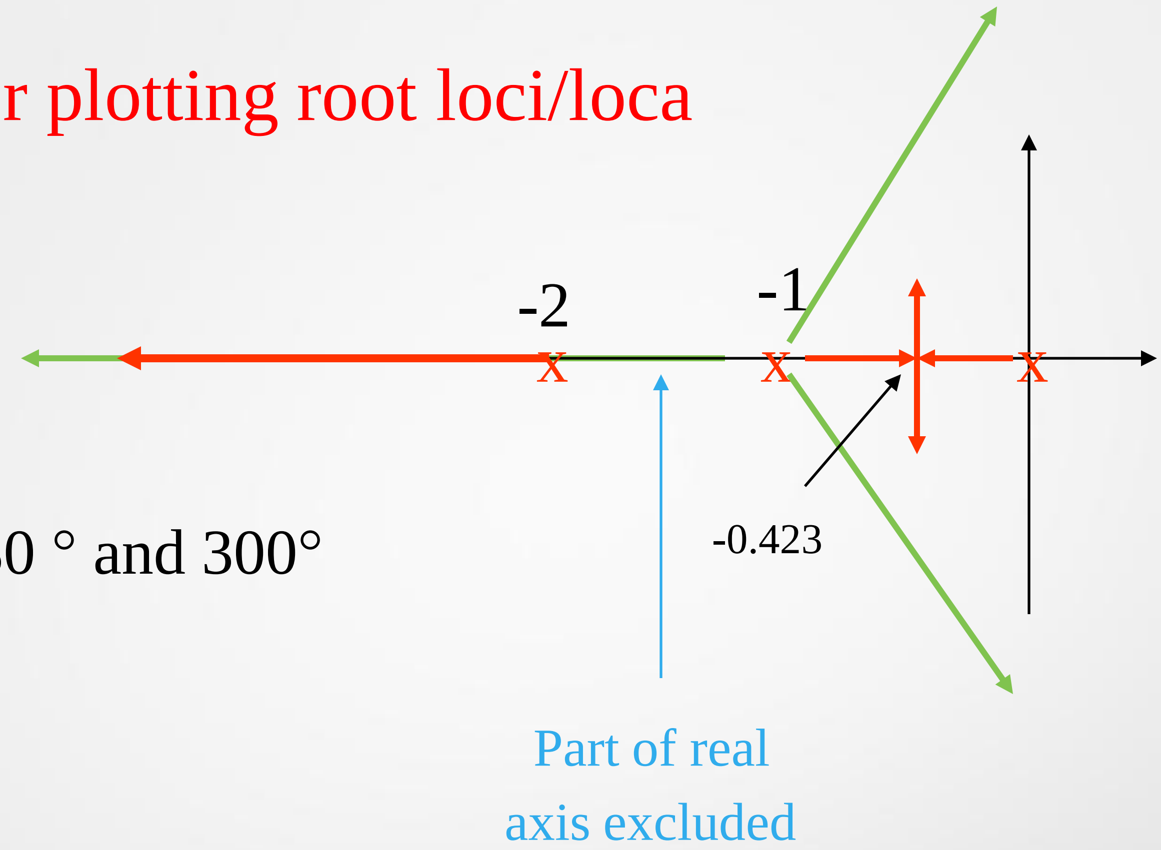
First method

$$\sum_{i=1}^{n} \frac{1}{\beta - p_i} = \sum_{i=1}^{m} \frac{1}{\beta - z_i}$$

Rules for plotting root loci/loca

Example

$$G(s)=\frac{K}{s(s+1)(s+2)}$$



3 asymptotes: 60 °,180 ° and 300°

$$\alpha=\frac{-1-2}{3}=-1$$

Break-away point

$$\frac{1}{\beta}+\frac{1}{\beta+1}+\frac{1}{\beta+2}=0$$

$$3\beta^2+6\beta+2=0$$

$$\beta=-0.423,-1.577$$

Rules for plotting root loci/loca

Second method

The break-away point is found by differentiating V(s) with Respect to s and equate to zero

$$v(s) = \frac{1}{G(s)}$$

Example $G(s) = \frac{K}{s(s+1)(s+2)}$

$$V(s) = \frac{s(s+1)(s+2)}{K} = \frac{s^3 + 3s^2 + 2s}{K}$$

$$\frac{dV}{ds} = \frac{3s^2 + 6s + 2}{K} = 0$$

$$s_\beta = -1.577, -0.423$$

Rules for plotting root loci/loca

- Rule 8: Intersection of root locus with the imaginary axis

The limiting value of K for instability may be found using the Routh criterion and hence the value of the loci at the Intersection with the imaginary axis is determined

Characteristic equation $$1 + K\frac{N(j\omega_1)}{D(j\omega_1)} = 0$$

Example $$G(s) = \frac{K}{s(s+1)(s+2)}$$

Characteristic equation $$s^3 + 3s^2 + 2s + K = 0$$

Rules for plotting root loci/loca

$$s^3 + 3s^2 + 2s + K = 0$$

What do we get with Routh ?

s^3	1	2
s^2	3	K
s^1	$6-K$	0
s^0	K	0

If K=6 then we have an pure imaginary solution

Rules for plotting root loci/loca

Example

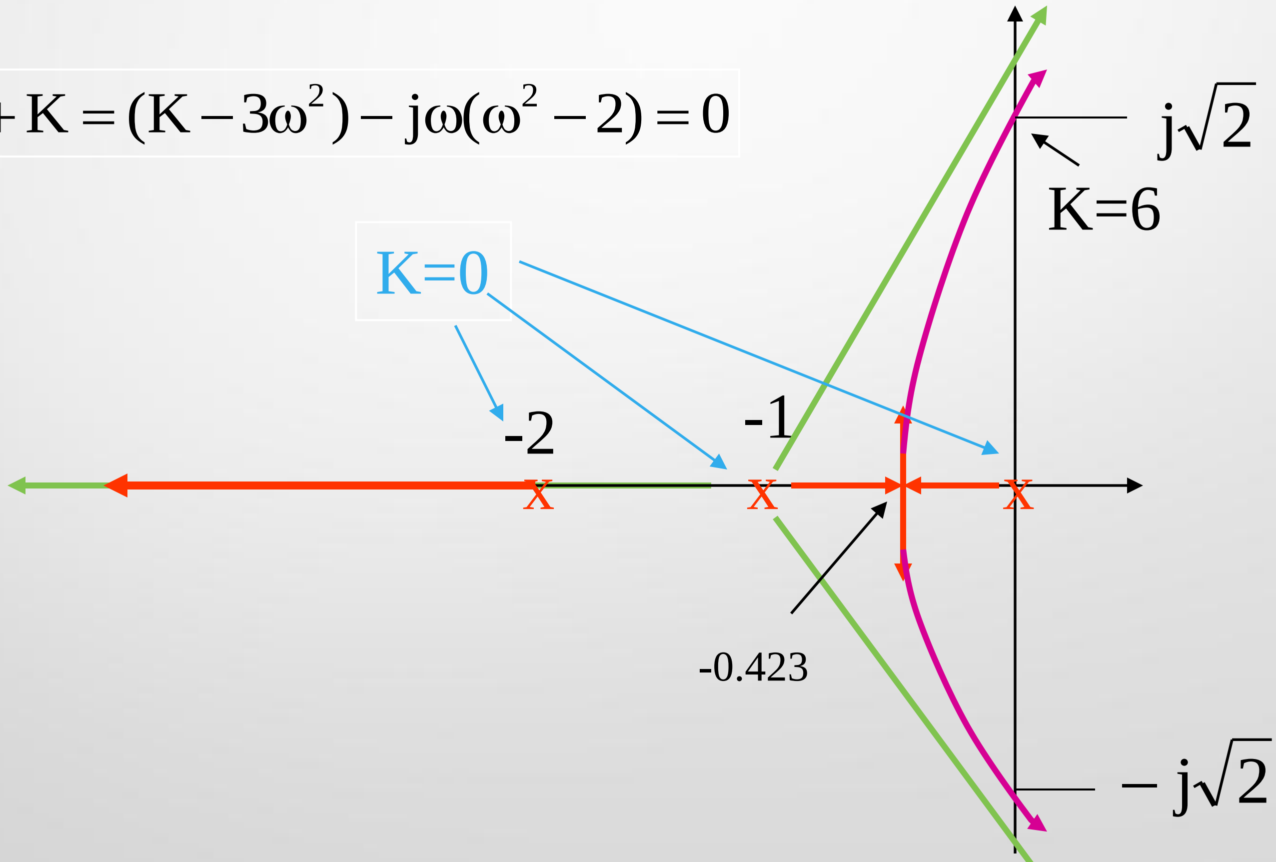
$$s^3 + 3s^2 + 2s + K = 0$$

$$s = j\omega$$

$$-j\omega^3 - 3\omega^2 + 2j\omega + K = (K - 3\omega^2) - j\omega(\omega^2 - 2) = 0$$

$$\omega^2 = 2$$

$$K = 3\omega^2 = 6$$



Rules for plotting root loci/loca

- Rule 9

Tangents to complex starting pole is given by

$$\Phi_s = 180^\circ - \arg(GH')$$

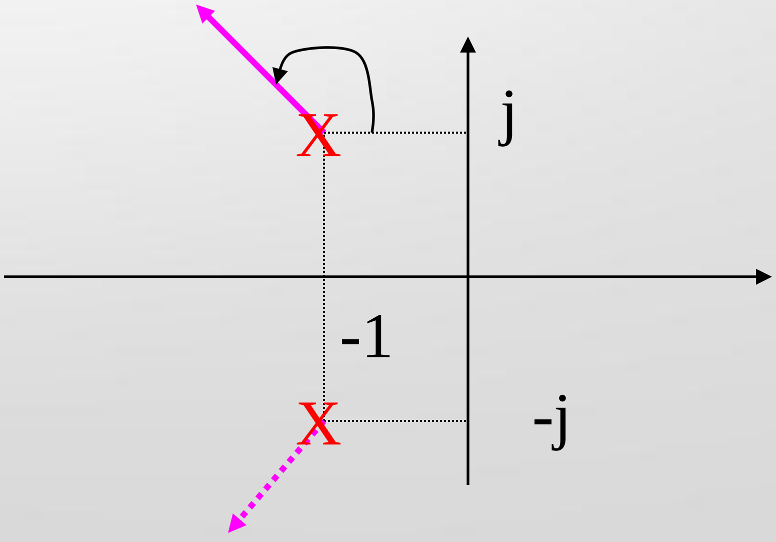
GH' is the GH(starting p) when removing starting p

Example

$$GH = \frac{K(s+2)}{(s+1+j)(s+1-j)}$$

$$GH'(s=-1+j) = \frac{K(1+j)}{(2j)}$$

$$\mathrm{Arg}\,GH' = 45^\circ - 90^\circ = -45^\circ$$



$$\Phi_s = 180^\circ - 45^\circ = 135^\circ$$

Rules for plotting root loci/loca

- Rule 9`

Tangents to complex terminal zero is given by

$$\Phi_T = 180^\circ - \arg(GH'')$$

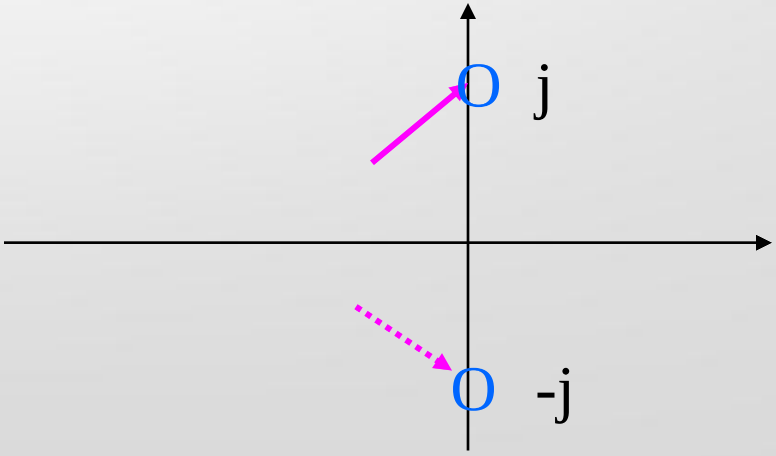
GH' is the GH(terminal zero) when removing terminal zero

Example

$$GH = \frac{K(s+j)(s-j)}{s(s+1)}$$

$$GH'(s=j) = \frac{K(2j)}{j(1+j)}$$

$$\mathrm{Arg}\,GH' = -45^\circ$$



$$\Phi_S = 180^\circ - (-45^\circ) = 225^\circ$$

Rules for plotting root loci/loca

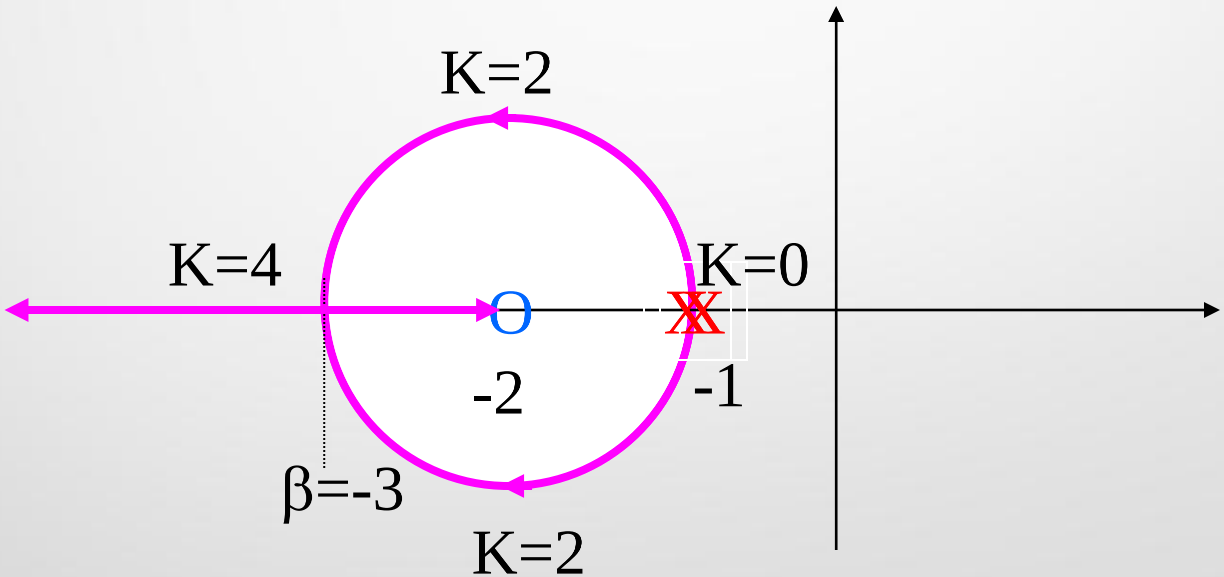
Example

$$GH(s) = \frac{K(s+2)}{(s+1)^2}$$

Two poles at –1
One zero at –2
One asymptote at 180°
Break-in point at -3

$$\frac{2}{\beta+1} = \frac{1}{\beta+2}$$

$$2\beta + 4 = \beta + 1$$



Rules for plotting root loci/loca

Example Why a circle ?

Characteristic equation $$s^2 + s(2+K) + 2K + 1 = 0$$

For K<4

$$s_{1,2} = \frac{-(2+K) \pm j\sqrt{K(4-K)}}{2}$$

For K>4

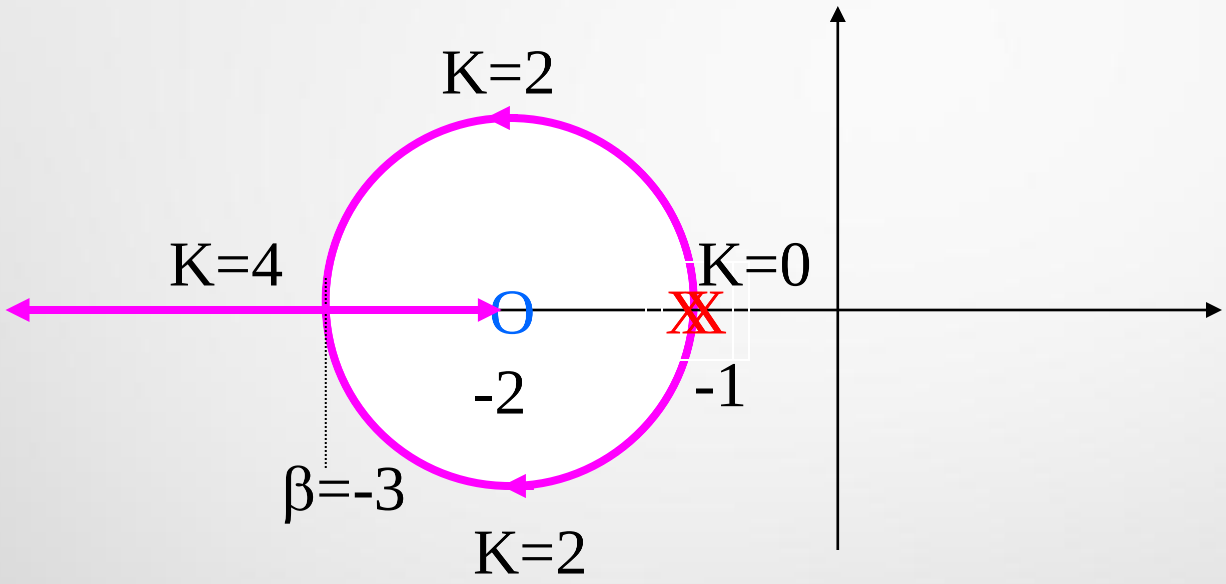
$$s_{1,2} = \frac{-(2+K) \pm \sqrt{K(K-4)}}{2}$$

Change of origin $$s_{1,2} + 2 = \frac{-(-2+K) \pm j\sqrt{K(4-K)}}{2}$$

$$4m = (K-2)^2 + K(4-K) = K^2 - 4K + 4 + 4K - K^2$$

$$m = 1$$

Rules for plotting root loci/loca



Root Locus Method

- The root locus is a powerful tool for designing and analyzing feedback control systems.
- It is possible to use root locus methods for design when two or three parameters vary. This provides us with the opportunity to design feedback systems with two or three adjustable parameters. For example the PID controller has three adjustable parameters.
- The root locus is the path of the roots of the characteristic equation traced out in the s-plane as a system parameter is changed.
- The design by the root locus method is based on reshaping the root locus of the system by adding poles and zeros to the system open loop transfer function and forcing the root loci to pass through desired closed-loop poles in the s-plane.

The root Locus Procedure

Step 1 : The characteristic equation $1+GH(s)=1+\dfrac{K\left(\frac{1}{2}s+1\right)}{s\left(\frac{1}{4}s+1\right)}=0$

Step 2 : The transfer function $GH(s)$ ***is written in terms of poles and zeros :*** $1+\dfrac{2K(s+2)}{s(s+4)}=0$

The multiplicative gain parameter is $2K$. ***To determine the locus of roots for the gain*** $0 \le K \le \infty$ ***(Step3) we locate the poles and zeros on the real axis.***

Step 4 : The angle criterion is satisfied on the real axis between the points 0 and - 2, because the angle p1 at the origin is 180^o***, and the angle from the zero and pole*** p_2 ***at*** $s = -4$ ***is zero degrees.***

The locus begins at the poles and ends at the zeros.

Step 5 : Find the number of separate loci (equal to the number of poles).

Step 6 : The root loci must be symmetrical with respect to the horizontal real axis.

Step 7 : The loci proceed to the zeros at infinity along asymptotes centered at σ_A ***and with angle*** ϕ_A.

Step 8 : Determine the point at which the locus crosses the imaginary axis.

Step 9 : Determine the breakway point on the real axis.

Step 10 : Determine the angle of departure of the locus from a pole and the angle of arrival at a zero.